Obstruction Certificates for Geometrically Defined Graph (and Digraph) Classes

Pavol Hell, Simon Fraser University

ATCAGC Durham, January 9, 2017

Plan

Emphasis on obstruction characterizations

- Interval graphs
- List homomorphisms
- Interval bigraphs and digraphs
- Bi-arc digraphs
- Circular arc graphs

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- Circular arc graphs

Mentioning joint work with

- Arash Rafiey
- Tomás Feder
- Jing Huang
- Juraj Stacho
- Mathew Francis

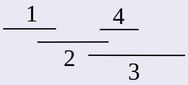
Interval Graphs

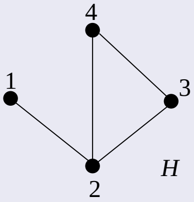
Interval graph

Vertices v can be represented by intervals I_v, so that

$$v \sim w \iff I_v \cap I_w \neq \emptyset$$

Example





Algorithms

$O(m + n)$ recognition algorithms

Booth-Lueker 1976, Korte-Mohring 1989, Habib-McConnell-Paul-Viennot 1998, Corneil-Olariu-Stewart 1998

Greedy $O(n)$ optimization algorithms

Gavril 1974, Rose-Tarjan-Lueker 1976

Interval Graphs

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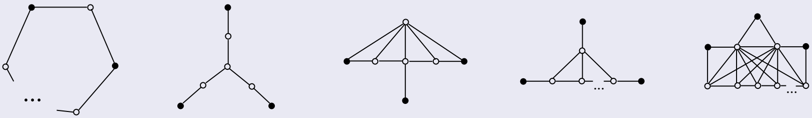
Applications

Food webs, resource allocation, genetics, etc.

Benzer 1959, Cohen 1978, Klee 1969

Lekkerkerker-Boland 1962

H is an interval graph $\iff$ H has no induced subgraph from



Interval Graphs

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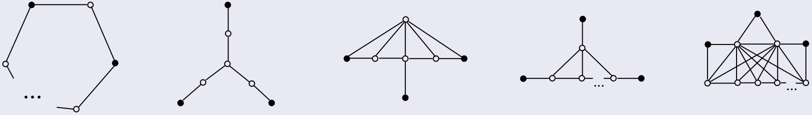

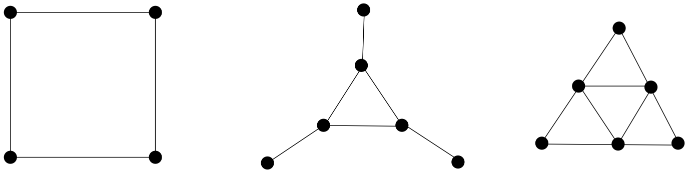

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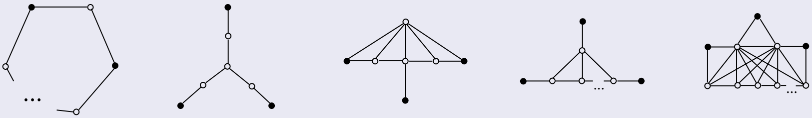

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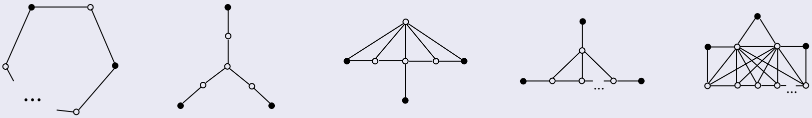
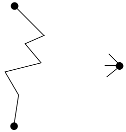


Asteroidal triple (AT)

Any two joined by a path avoiding the neighbours of the third



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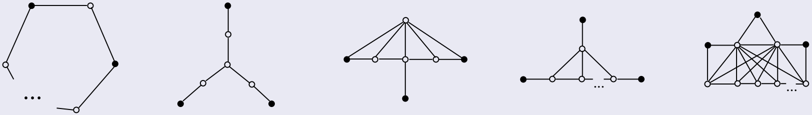

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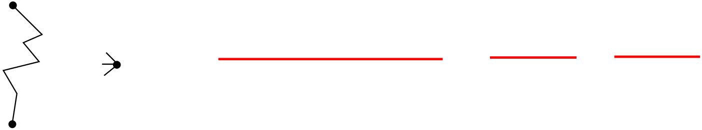

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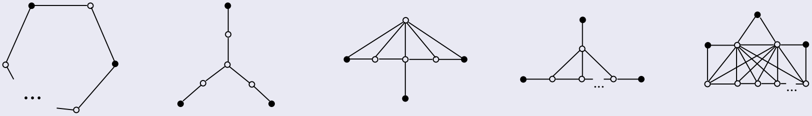

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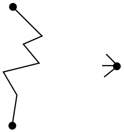

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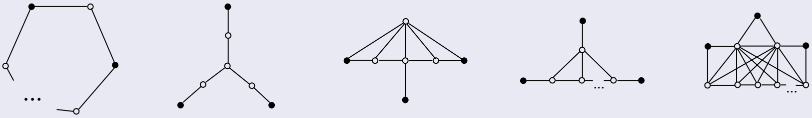
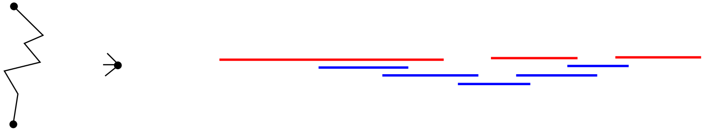
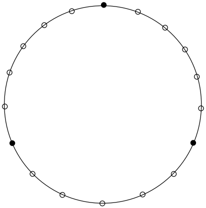


Lekkerkerker-Boland 1962

H is an interval graph $\iff$ H has no AT or induced C_k, $k \geq 4$.



A Structural Characterization



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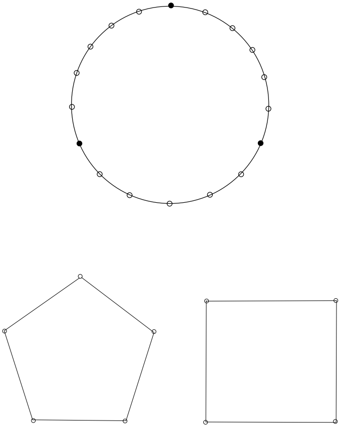
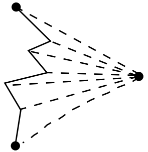
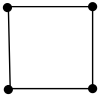


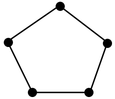
A Structural Characterization

Lekkerkerker-Boland 1962

H is an interval graph $\iff$ H has no AT or induced C_4, C_5.







An Ordering Characterization

Min-ordering

H is an interval graph

$$\iff$$

$V(H)$ can be linearly ordered by $<$ so that

$$u \sim v, u' \sim v' \text{ and } u < u', v' < v \implies u \sim v'$$

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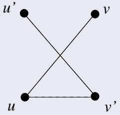
Dotted edge cannot be absent



An Ordering Characterization

Min-ordering

H is an interval graph

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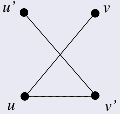
$V(H)$ can be linearly ordered by $<$ so that

$$u \sim v, u' \sim v' \implies \min(u, u') \sim \min(v, v')$$

Dotted edge cannot be absent



An Ordering Characterization

Min-ordering

H is an interval graph

$$\iff$$

H has a *min ordering*, i.e.,
$V(H)$ can be linearly ordered by $<$ so that

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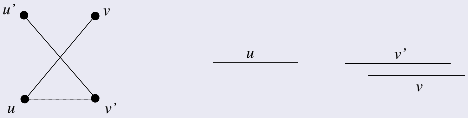
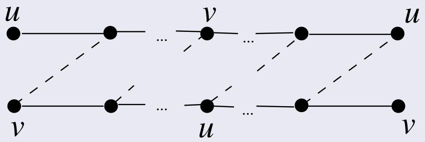
Proof of $\implies$: order by left endpoints



An Obstruction to Min Ordering

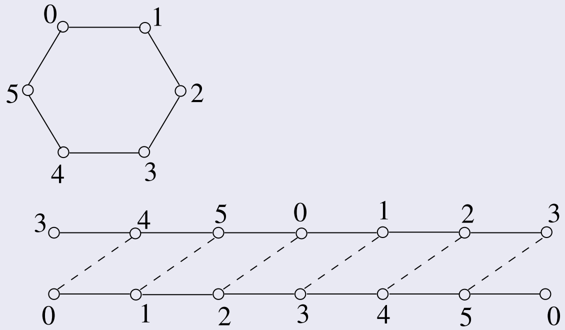
Invertible pair



Dashed line = non-edge

An Obstruction to Min Ordering

A graph with an invertible pair cannot have a min ordering



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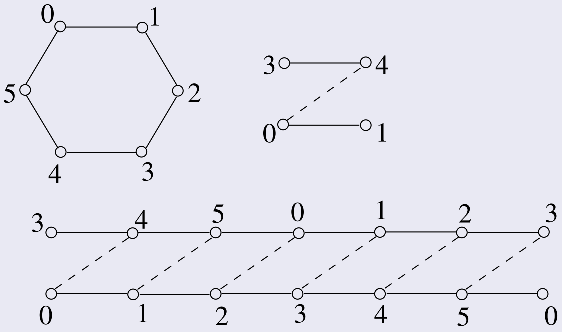

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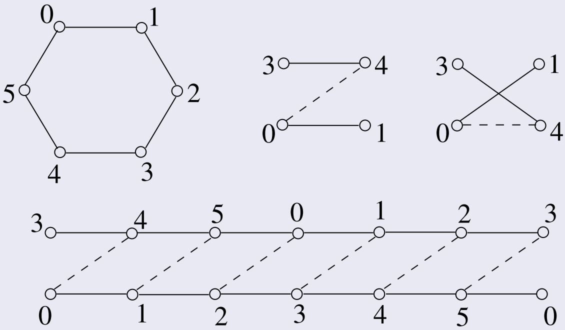
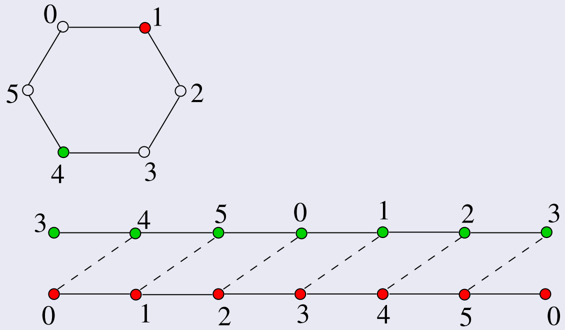


An Obstruction to Min Ordering

C_6 has an invertible pair



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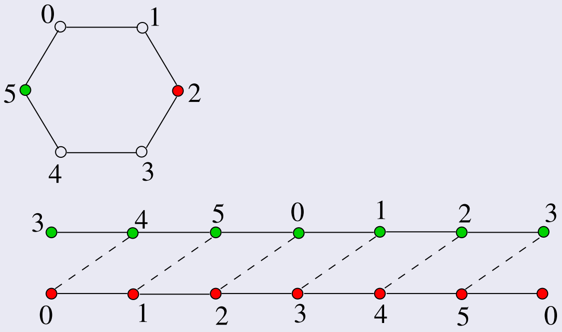

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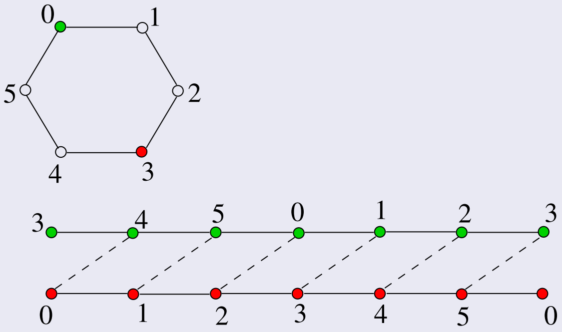

Obstruction Theorems

The following statements are equivalent

1. G is an interval graph
2. G has a min ordering
3. G has no invertible pair
4. G has no AT or induced C_4 or C_5

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1. G is an interval graph
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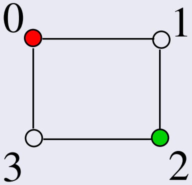
Shown: $1 \implies 2$, $2 \implies 3$, and $4 \implies 1$

To show: $3 \implies 4$

$3 \implies 4$

Cycles C_4, C_5 and all AT have an invertible pair



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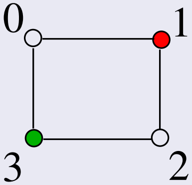

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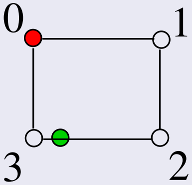
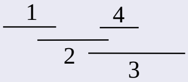
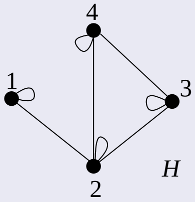


Observation

Interval graphs are reflexive (have all loops)





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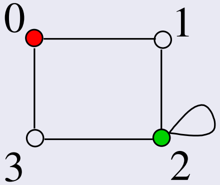

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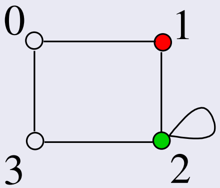

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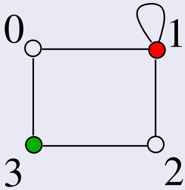

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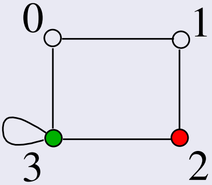

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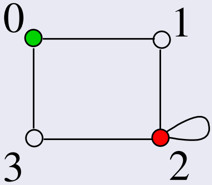

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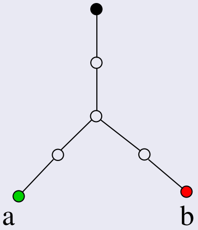

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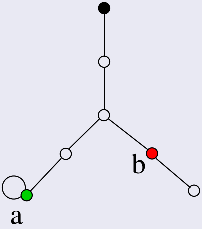

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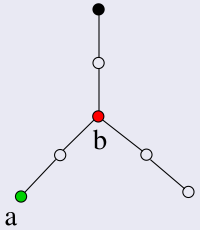

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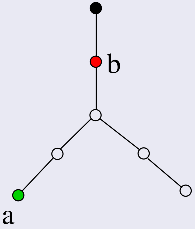

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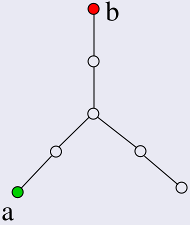

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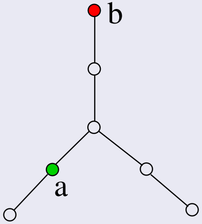

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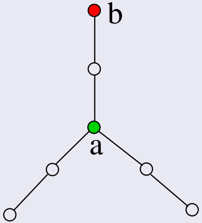

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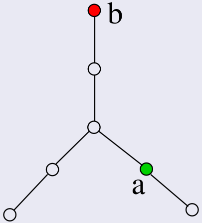

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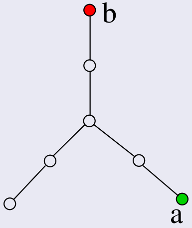

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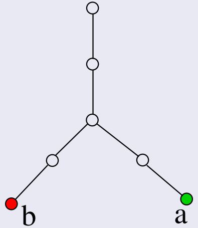
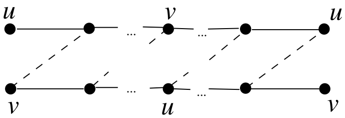


The Pulsed Obstruction Theorem

A New Characterization

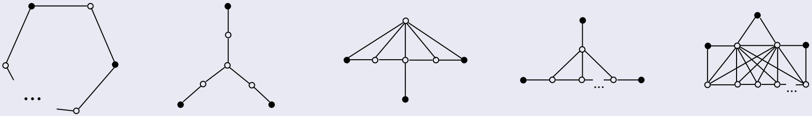
H is an interval graph $\iff$ it has no invertible pair

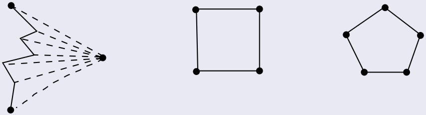


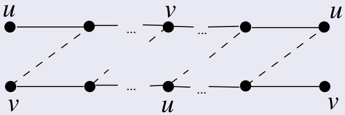
Feder+H+Huang+Rafiey 2012

Obstructions to Interval Graphs

Three related characterizations







Applications

Qui a tué le Duc de Densmore?

- Anne met Felicia, Cynthia, Georgia, Emilie, and Betty
- Betty met Cynthia, Anne, and Helen
- Cynthia met Anne, Emily, Diane, Betty, and Helen
- Diane met Cynthia and Emily
- Emily met Felicia, Cynthia, Diane, and Anne
- Felicia met Emily and Anne
- Georgia met Anne and Helen
- Helen met Cynthia, Georgia, and Betty

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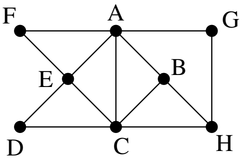

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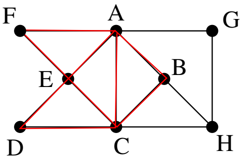

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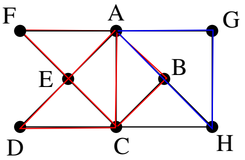

Applications

Qui a tué le Duc de Densmore?

- Anne met Felicia, Cynthia, Georgia, Emilie, and Betty
- Betty met Cynthia, Anne, and Helen
- Cynthia met Anne, Emily, Diane, Betty, and Helen
- Diane met Cynthia and Emily
- Emily met Felicia, Cynthia, Diane, and Anne
- Felicia met Emily and Anne
- Georgia met Anne and Helen
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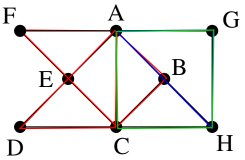

Applications

$O(m + n)$ certifying recognition algorithm

Either produces an interval representation or an AT or an induced cycle > 3

Kratsch-McConnell-Mehlhorn-Spinrad 2006

Applications

The list homomorphism problem for a graph H

Given a graph G with lists $L(v) \subseteq V(H), v \in V(G)$,
is there a homomorphism $f : G \to H$ (vertex-mapping with $u \sim v \implies f(u) \sim f(v)$)
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If H is has a min ordering then the list homomorphism problem for H admits a polynomial time algorithm

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Dichotomy for reflexive graphs H

If H is an interval graph then the list homomorphism problem for H admits a polynomial time algorithm, otherwise it is NP-complete

Feder-H 1998

Bigraphs

Bipartite graphs with red vs blue vertices

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Bipartite graphs with red vs blue vertices

Interval bigraph

Representable by real intervals I_r, J_b (for r red and b blue)

$$r \sim b \Longleftrightarrow I_r \cap J_b \neq \emptyset$$

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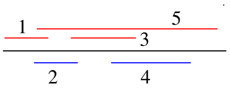

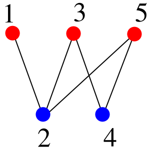

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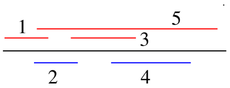

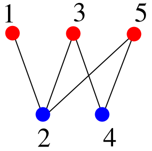

Sen-Das-Roy-West 1989

No obstruction characterizations, recognition $O(n^{15})$ Mueller 1997

Faster algorithms claimed by Rafiey 2013 and by Das 2013

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Min ordering of a bigraph H

A linear ordering $<$ of $V(H)$ so that

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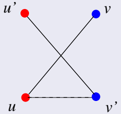

Bigraphs

Two geometric representations

H has a min ordering $\iff$ $\overline{H}$ is a circular arc graph

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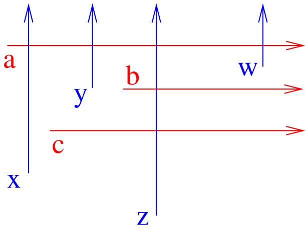
H has a min ordering $\iff$ H is a 2-directional ray graph

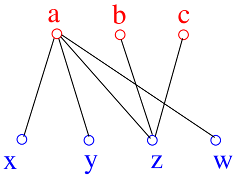
Feder, H and Huang 1999

Shrestha, Tayu, and Ueno 2010, H+Rafiey 2011

A 2DR graph

Intersection graph of a family of UP and RIGHT rays

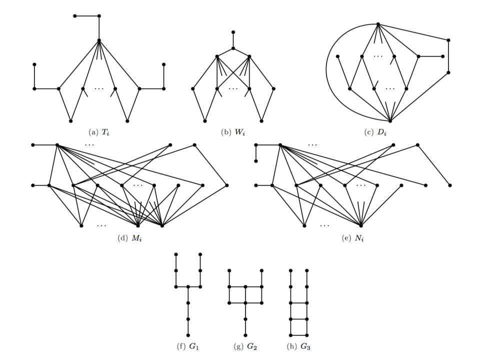




Obstruction Characterizations

A bigraph H is a 2DR graph $\iff$

- does not contain an induced cycle or any subgraph from

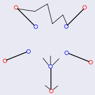


Trotter and Moore 1976

Obstruction Characterizations

A bigraph H is a 2DR graph $\iff$

- does not contain an induced subgraph from the list
- H has no induced $C_{>4}$ and no edge-asteroid



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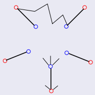
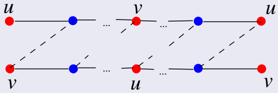


- H has no invertible pair



Trotter and Moore 1976; H and Huang 2004; H and Rafiey 2011; Shrestha, Tayu, and Ueno 2010

Two Directional Ray Graphs

Similarities to interval graphs

- similar geometric representations
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Hell-Rafiey

$O(n^2)$ recognition

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Open

An $O(m + n)$ recognition algorithm?

Two Directional Ray Graphs

2DR graphs are a better analogue of interval graphs than interval bigraphs

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2DR graphs are more general than interval bigraphs

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H and Huang 2004

Digraphs

An interval digraph

Vertices can be represented by pairs of intervals I_v, J_v, so that

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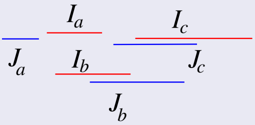
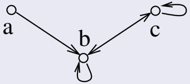
Example





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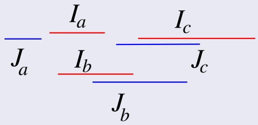

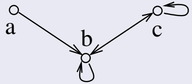

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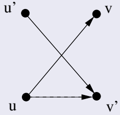

Reflexive Digraphs

A geometric representation

A reflexive digraph has has a min ordering $\iff$ it is an adjusted interval digraph

Feder+H+Huang+Rafiey 2012

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Adjusted interval digraphs

Vertices can be represented by pairs of *adjusted* intervals I_v, J_v, so that

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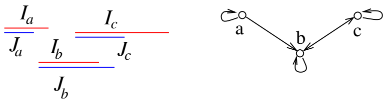

Obstruction Characterization

A reflexive digraph H an adjusted interval digraph if and only if

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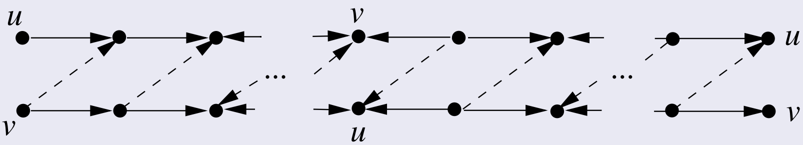

Adjusted Interval diraphs

Similarities to interval graphs

- similar geometric representations
- similar obstructions
- similar ordering characterization
- similar polynomial time algorithm for the list homomorphism problem to an adjusted interval digraph H

$O(n^4)$ recognition algorithm

Open

A more efficient recognition algorithm?

Another similarity

Dichromatic number of H

The minimum number of acyclic parts H can be partitioned into

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The minimum number of acyclic parts H can be partitioned into

H is an adjusted interval digraph (without the loops)

Linear time algorithm for the dichromatic number

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Each directed cycle in an adjusted interval digraph contains a digon

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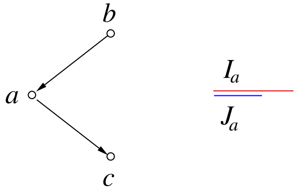

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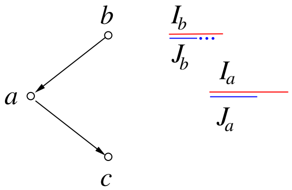

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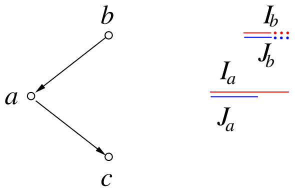

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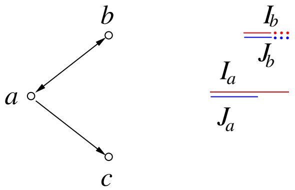
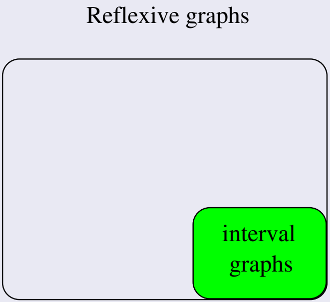


The World of Digraphs

Interval-like graphs



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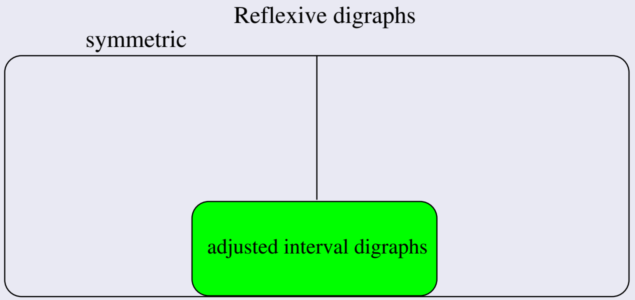

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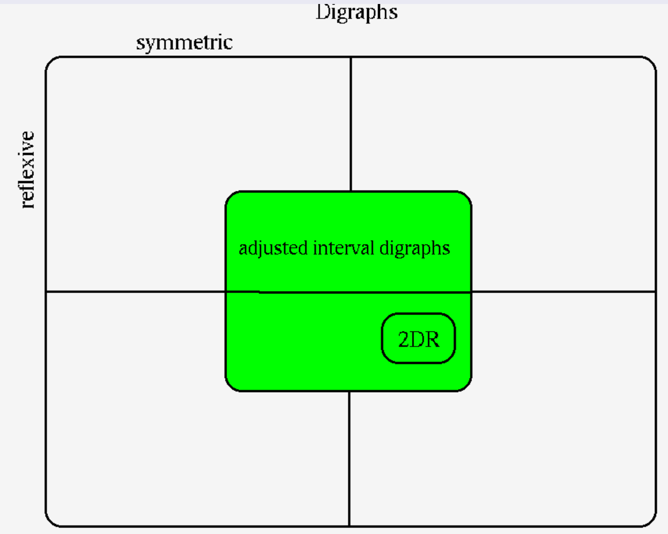

Interval-like digraphs

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Min-orderable digraphs?

The World of Digraphs

Interval-like digraphs

Min-orderable digraphs?

- Geometric representation?
- Obstruction characterization?
- Polynomial recognition algorithm?

General Digraphs

The following are equivalent

- H has a min ordering
- H is a bi-arc digraph
- H has no invertible circuit (testable in $O(n^4)$)

H+Rafiey 2016

Bi-Arc Digraphs

A bi-arc digraph H

Representable by two *consistent* families of circular arcs

$$I_v, v \in V(H), \text{ and } J_v, v \in V(H),$$

$$uv \in E(H) \iff I_u \cap J_v = \emptyset$$

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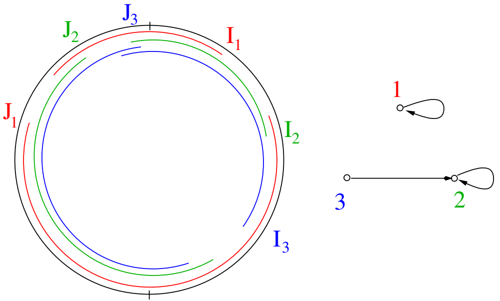

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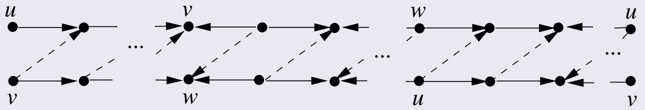
Invertible circuit



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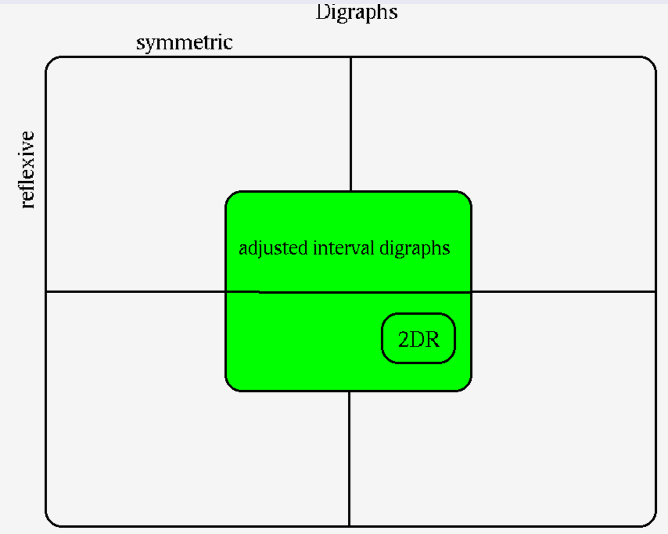
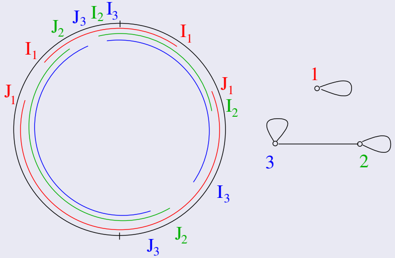


Special cases of bi-arc digraphs

Bi-Arc Digraphs

Special cases of bi-arc digraphs

- reflexive and symmetric digraph $\Longleftrightarrow$ interval graph



Same order: 1, 1, 2, 3, 2, 3

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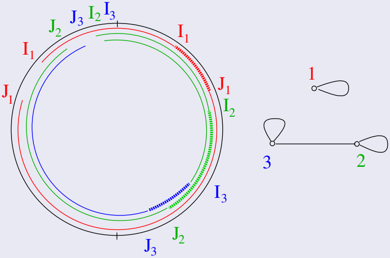

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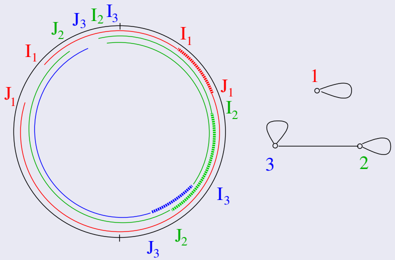

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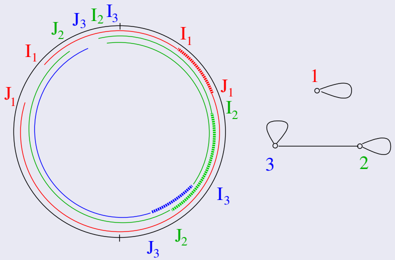

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- reflexive digraph $\iff$ adjusted interval digraph
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Dichotomy Classifications

The list homomorphism problem for H

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Bulatov 2005

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Difficulties with circular arc graphs

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- May have exponentially many maxcliques
- Not all perfect

Circular Arc Graphs

Recognition algorithms

Circular Arc Graphs

Recognition algorithms

- $O(n^3)$ Tucker 1980

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Certifying algorithm?

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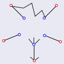
For a co-bipartite graph H

H circular arc $\iff$ $\overline{H}$ has no induced $C_{>4}$ and no edge-asteroid



A Pulsed Obstruction Characterization

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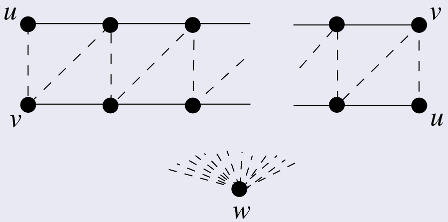
An anchored invertible pair



Francis+H+Stacho 2015

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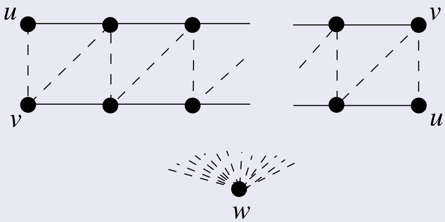
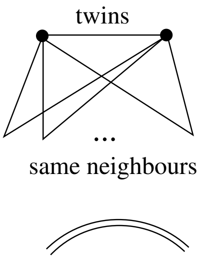


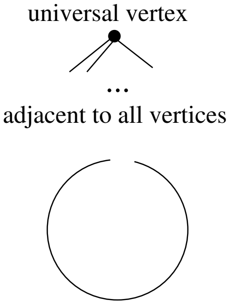
Francis+H+Stacho 2015

UNDER THE RIGHT INTERPRETATION AND ASSUMPTIONS

The First Twist – Standard

H has no twins and universal vertices



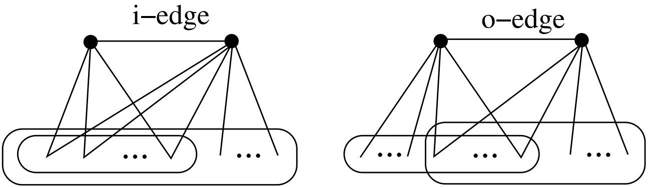


The First Twist – Standard

Each edge of H has a "type"

Type of edge uv

- Type i if $N[u] \subseteq N[v]$ ("inclusion")
- Type o if each u, v has a private neighbour ("overlap")



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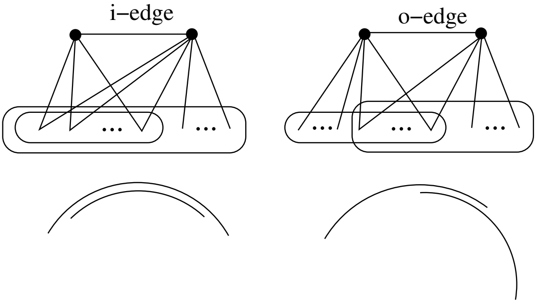

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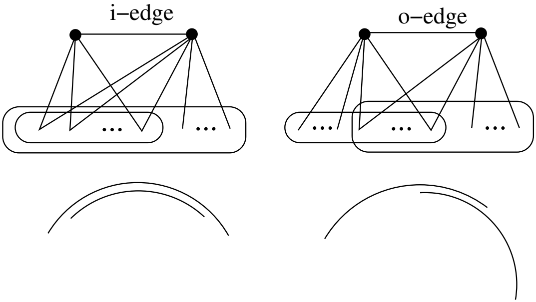

Hsu 1995

If H has a circular arc representation, then it has one corresponding to the labels

The Second Twist – New

Extend H to include "complements"

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Circularly paired vertices u, v

- u and v are not adjacent
- $x \not\sim u \implies xv$ is an i-edge, and
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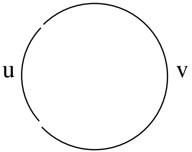

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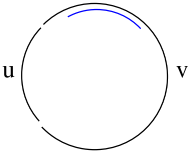

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If u is not circularly paired in H, we add a suitable new vertex $\overline{u}$ $(x \sim \overline{u} \iff xu$ is not an i-edge)

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$(x \sim \overline{u} \iff xu$ is not an i-edge$)$

Facts

Each H has a unique circular completion H^+
H is a circular arc graph $\iff H^+$ is a circular arc graph

The Structural Characterization

Review all assumptions

- H has no twins and no universal vertices
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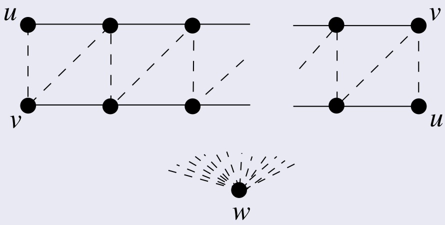
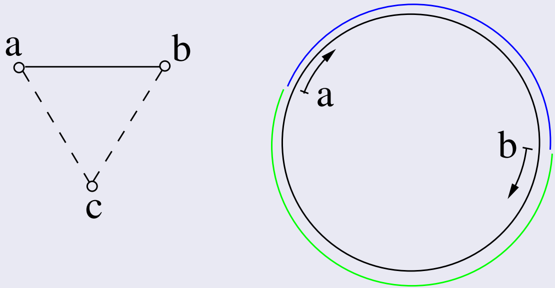
Obstruction to circular arc graphs



Delta Triangles

If it could be represented



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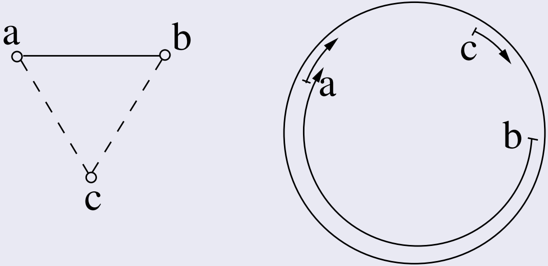

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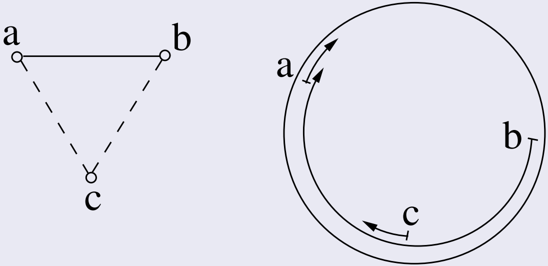

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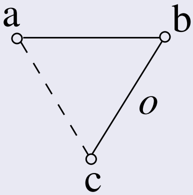

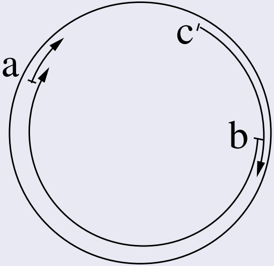

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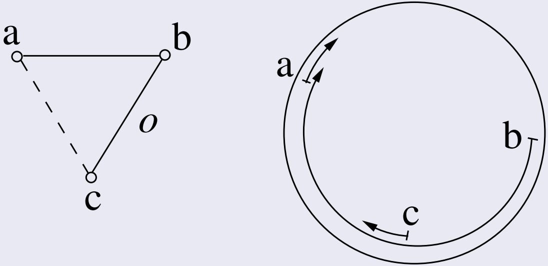

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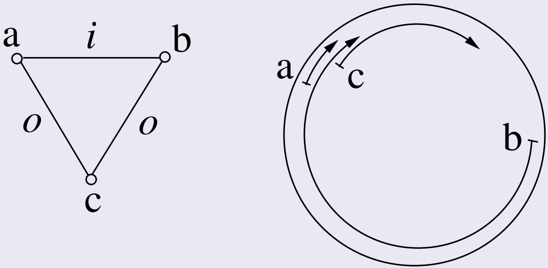

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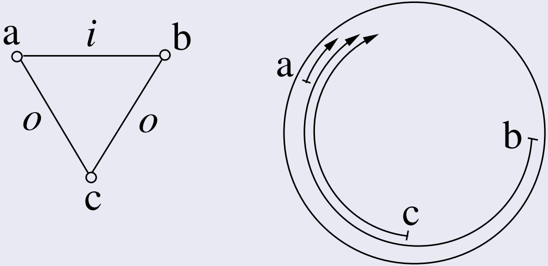
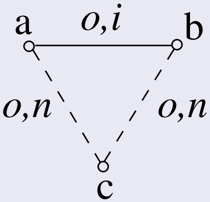
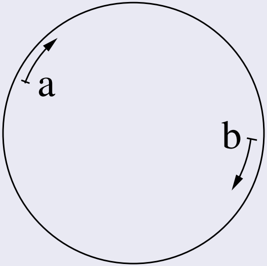


c must be on opposite side of where a meets b





NOT ALL o

Necessity

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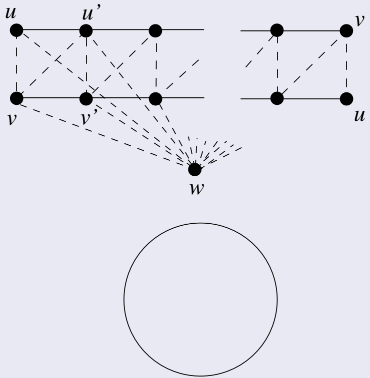

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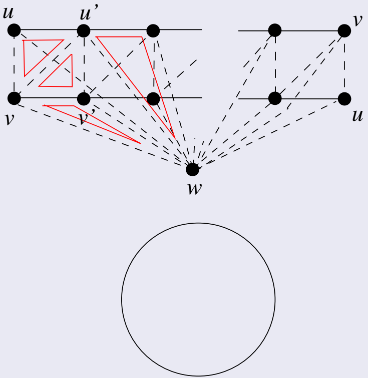

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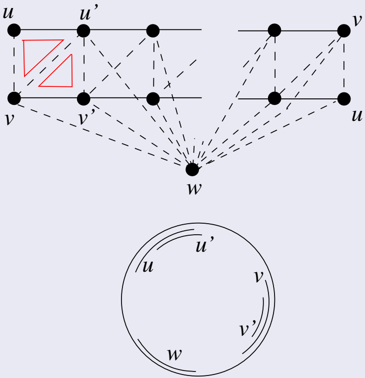
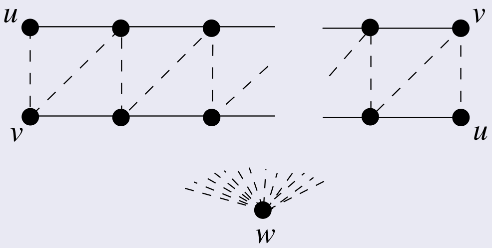


The Structural Characterization

Anchored invertible pair



Dashed line = non-edge or o-edge

Each triangle with a horizontal edge is a delta triangle

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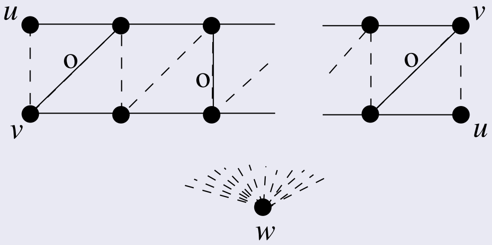

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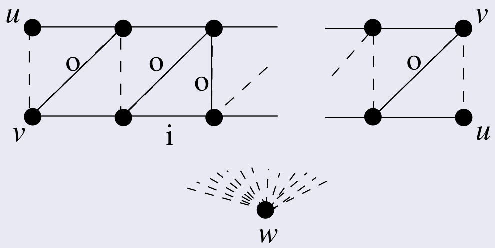

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The Structural Characterization

Assumptions

- H has no twins and no universal vertices
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Theorem

H is a circular arc graph $\iff$ it has no anchored invertible pair

Francis+H+Stacho 2015

A Certifying Algorithm

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Producing an anchored invertible pair

- Delete universal vertices

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- Run a standard recognition algorithm
  If a represenation is found, it is the certificate
  If no representation is found
  - Compute the edge-labels
  - Compute the circular completion
  - Find an anchored invertible pair (via an auxiliary graph)