Cubic graphs that cannot be covered with four perfect matchings

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Perfect matching covers of cubic graphs

Theorem (Petersen, 1891)

Every bridgeless cubic graphs contains a perfect matching.

Theorem (Schönberger, 1934)

Every edge of a bridgeless cubic graphs is contained in a perfect matching.

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Definition

The perfect-matching index of a bridgeless cubic graph G

$\tau(G) \quad = \quad$ the smallest # of perfect matchings that cover $E(G)$.

Bounds for perfect matching index

Observation

- $\tau(G) \geq 3$ for every bridgeless cubic graph G
- $\tau(G) = 3 \iff G$ is 3-edge-colourable.

Perfect matching index is only interesting for cubic graphs that have no 3-edge-colouring – that is, snarks.

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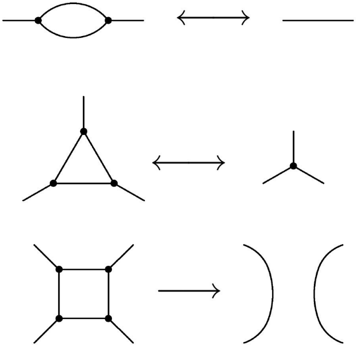
Definition

A snark is a cubic graph that has no 3-edge-colouring.

Snarks



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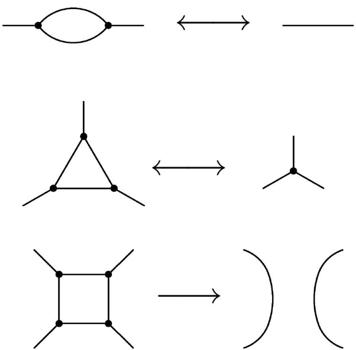

Similar simplifications for cycle-separating edge-cuts of size ≤ 3

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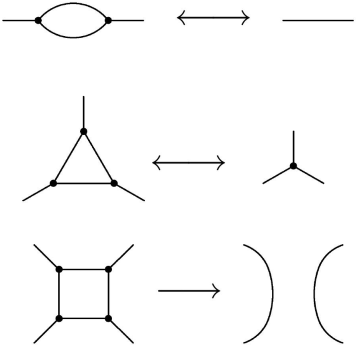

Similar simplifications for cycle-separating edge-cuts of size ≤ 3

$\Longrightarrow$ 'nontrivial' usually means

- cyclically 4-edge-connected, and
- girth > 4

Snarks

A 3-edge-colouring of a cubic graph G can be thought of as a mapping

$$\phi\colon E(G) \to \mathbb{Z}_2 \times \mathbb{Z}_2 - 0 = \{01, 10, 11\}$$

such that the sum of colours around each vertex $=$ 0.

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Snarks are cubic graphs that do not have a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow

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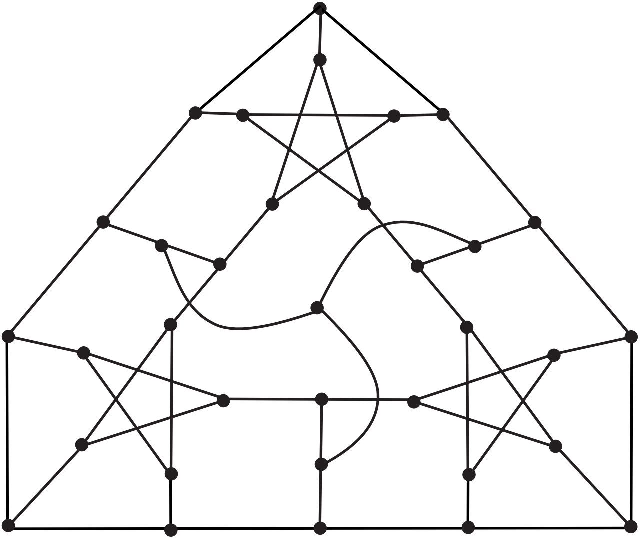
- There are exactly 64326024 such snarks with ≤ 36 vertices.
- Only two of them have $\tau(G) \neq 4$:
  the Petersen graph and a snark of order 34.

A snark of order 34 with $\tau(G) = 5$



Snarks with $\tau(G) \geq 5$: Construction 1

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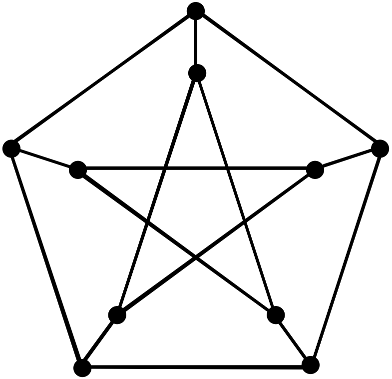

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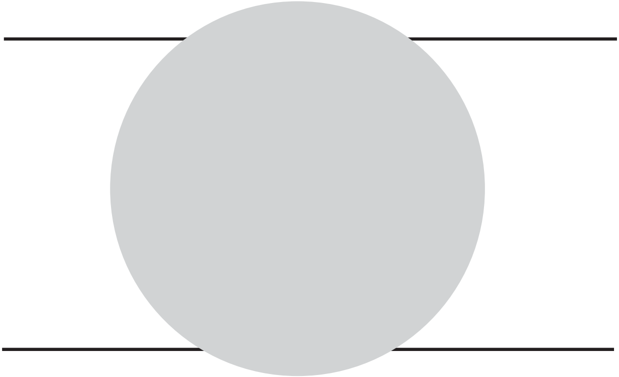

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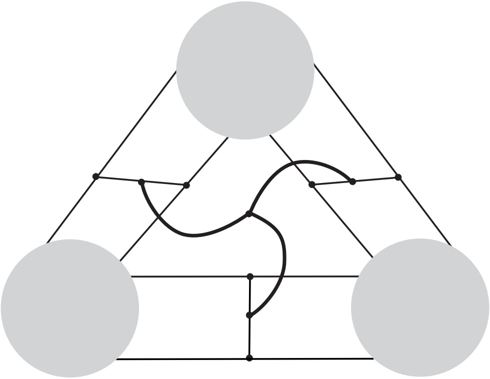

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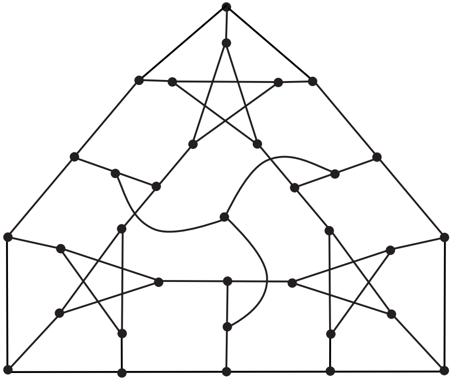

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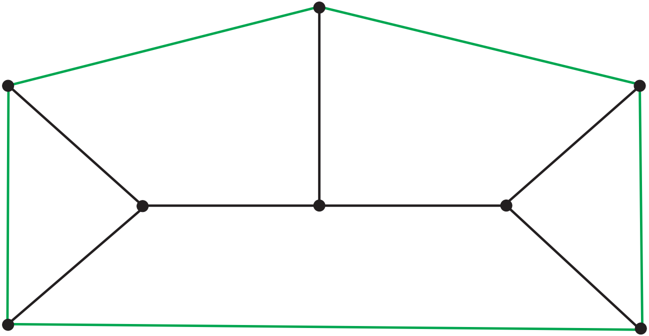

Halin graph

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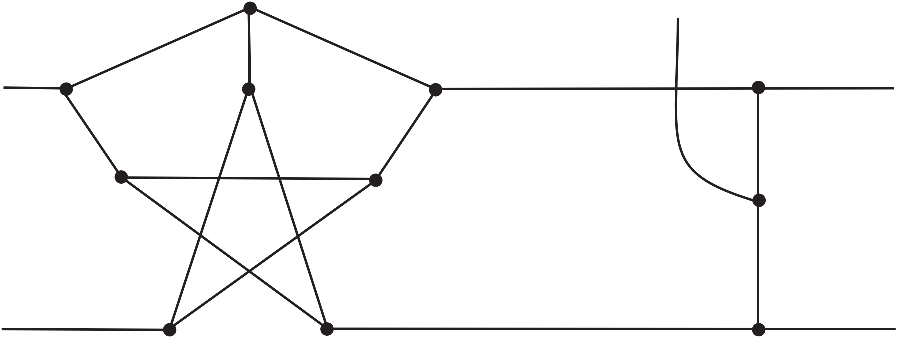

Petersen fragment

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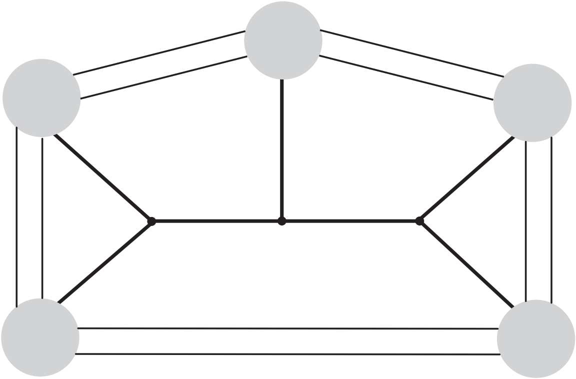

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- Treelike snarks have a more general shape than windmill snarks.

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- Treelike snarks have a more general shape than windmill snarks.

However:

- Building blocks are restricted to the Petersen graph.
- Proofs heavily depend on computer-aided arguments.

Snarks with $\tau(G) \geq 5$: Construction 2

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8.1 The pattern set of the Petersen fragment

The pattern set of F_0 (42 patterns):

A A AB AC AD	A B CD AB AB	A BC D BC BC
A A AB C D	A B CD AC AC	A BC D BD BD
A AB A AC AD	A B CD C C	A BC D D D
A AB A BC BD	A B CD CD CD	AB AB AB AC AD
A AB AC A AD	A BC A AB BD	AB AC AB AB AD
A AB AC B BD	A BC B AB AD	AB AC AB BC CD
A AB AC C CD	A BC B BC CD	AB AC AD A A
A B AB AB CD	A BC BD A AB	AB AC AD AB AB
A B AB AC BD	A BC BD BC C	AB AC AD AD AD
A B AC A D	A BC BD BD D	AB AC AD B B
A B AC AB BD	A BC D A A	AB AC AD BC BC
A B C A AD	A BC D AB AB	AB AC AD BD BD
A B C C CD	A BC D AD AD	AB AC AD D D
A B CD A A	A BC D B B	AB CD AC AB BC

Generalisation of both constructions: Halin snarks

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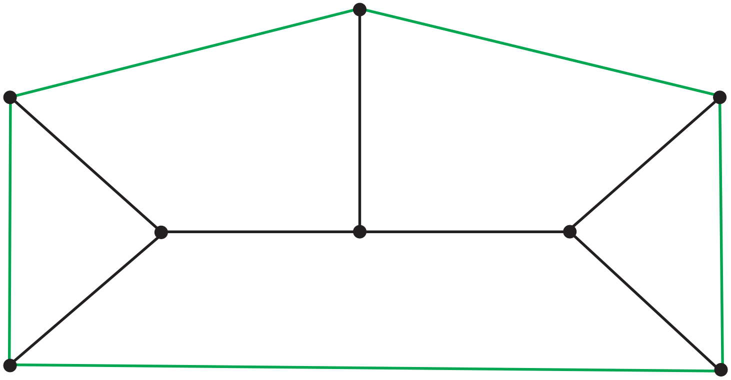

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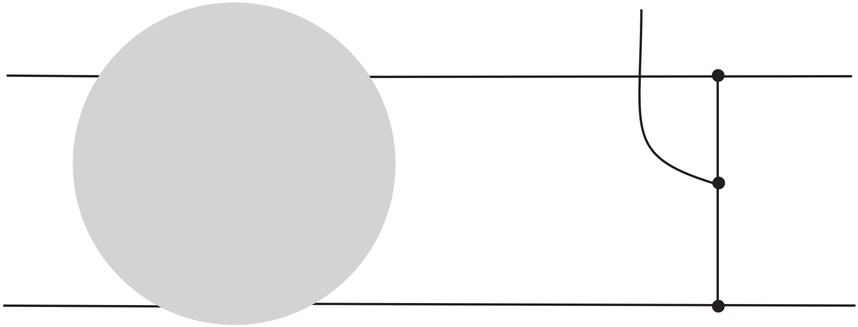

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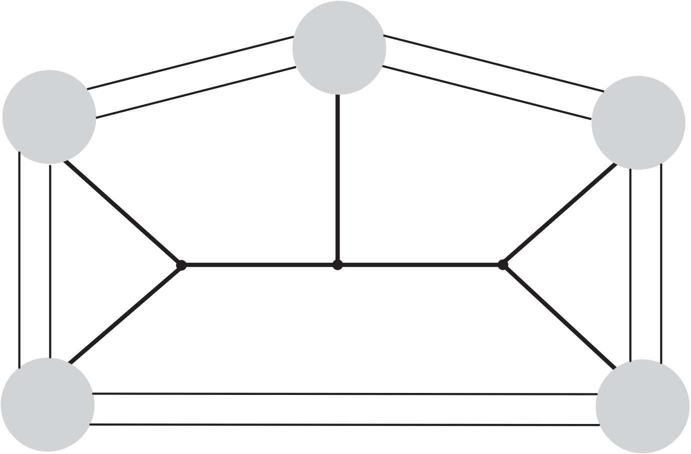

Halin snark

Perfect matching index of Halin snarks

Theorem (Máčajová & S., 2017+)

The perfect matching index of every Halin snark is at least 5.

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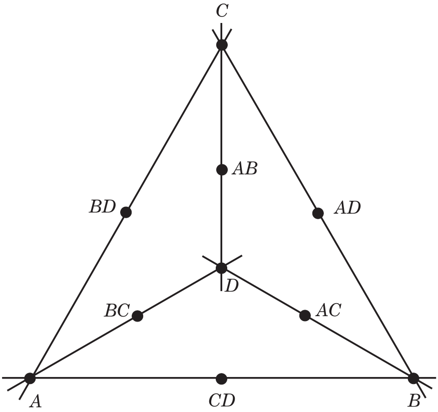
To prove the result we need two main ingredients:

- Characterisation of cubic graphs with $\tau(G) \leq 4$.
- Transition relation for patterns of perfect matching 4-covers for the building blocks of Halin snarks.

Characterisation of cubic graphs with $\tau(G) \leq 4$

Theorem

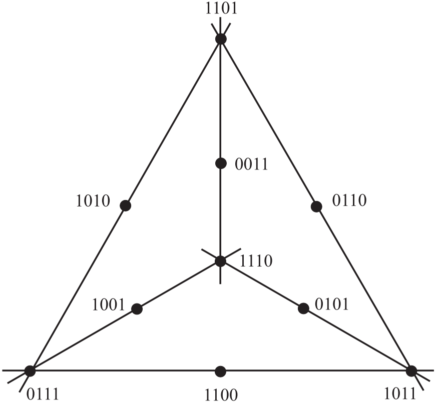
A cubic graph G can have its edges covered with four perfect matchings $\iff$ G has a proper edge-colouring with ten colours arranged into six 3-element blocks s.t. the colours around each vertex form a block.



Characterisation of cubic graphs with $\tau(G) \leq 4$

Theorem

In every cubic graph, there is a 1-1 correspondence between perfect matching 4-covers and edge-colourings with points of a configuration $\mathcal{T}$ of six lines of $PG(3,2)$ spanned by four points in general position.



Colourings by projective configurations

Let $\mathcal{C} = (P, L)$ be a configuration of points and lines in the n-dimensional projective space $GP(n, 2)$, $n \geq 1$, over the 2-element field.

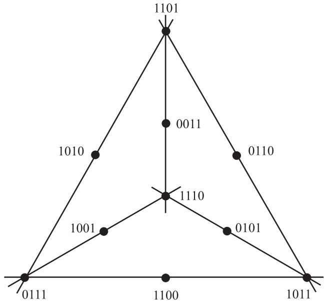
- points ... subset $P \subseteq \mathbb{Z}_2^{n+1} - 0$
- lines ... 3-element subsets $\{x, y, z\}$ of P with $x + y + z = 0$

Definition

A $\mathcal{C}$-colouring of a cubic graph G is $\phi \colon E(G) \to P$ such that for each vertex v of G the three colours around v form a line of $\mathcal{C}$.

Since the colour patterns around vertices are lines in $GP(n, 2)$
$\Longrightarrow$ every $\mathcal{C}$-colouring is a nowhere-zero $\mathbb{Z}_2^{n+1}$-flow ... $\mathcal{C}$-flow.

Colourings by projective configurations



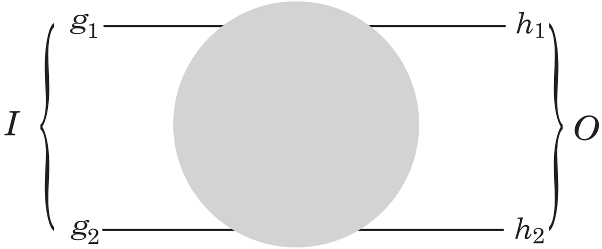
- $\mathcal{C}$-flows for $\mathcal{C} = \mathcal{T}$, the tetrahedral configuration of six lines in $PG(3,2)$, represent perfect matching 4-covers.
- $\mathcal{C}$-flows for $\mathcal{C} = \mathcal{I}$, the trivial configuration of a single line in $PG(1,2)$, represent nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flows, i.e., 3-edge-colourings

Transition relation for $(2, 2)$-poles

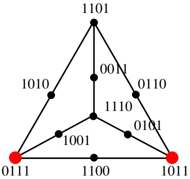
Definition

A dipole $D = D(I, O)$ admits a transition $\{x, y\} \to \{x', y'\}$, if there exists a $\mathcal{T}$-flow ξ on D such that

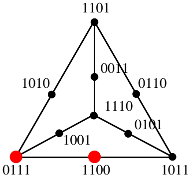
- $\{\xi(g_1), \xi(g_2)\} = \{x, y\}$, and
- $\{\xi(h_1), \xi(h_2)\} = \{x', y'\}$.



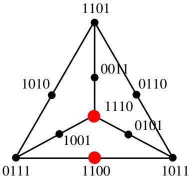
Pairs of points in configuration $\mathcal{T}$



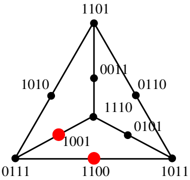
edge



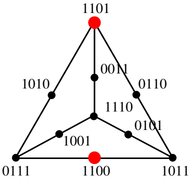
half–line



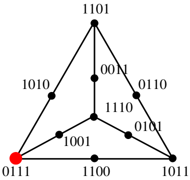
altitude



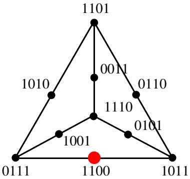
angle



axis



zero



zero

Dipoles obtained from graphs with $\tau \geq 5$

Lemma

Let $\{x, y\} \to \{x', y'\}$ be a transition through a dipole $X = X(I, O)$ obtained from cubic graph with $\tau \geq 5$ by removing two adjacent vertices. Then at most one of the pairs $\{x, y\}$ and $\{x', y'\}$ is collinear in $\mathcal{T}$.

Let us call such a dipole a dislineator.

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Theorem

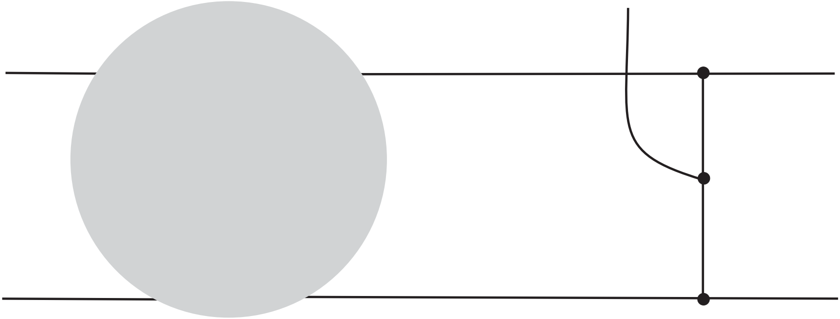
The following statements are equivalent for a dipole X:

- *X is a dislineator.*
- *Adding two adjacent vertices to X and attaching each of them to a connector of X produces a cubic graph with $\tau \geq 5$.*
- *X has no transitions of type* `edge` $\to$ `edge` *or* `half-line` $\to$ `half-line`.

Transition relation for general fragments

A general fragment is obtained from an arbitrary dislineator by attaching a $(2,2;1)$-pole on three vertices.

It has input, output, and a residual semiedge.



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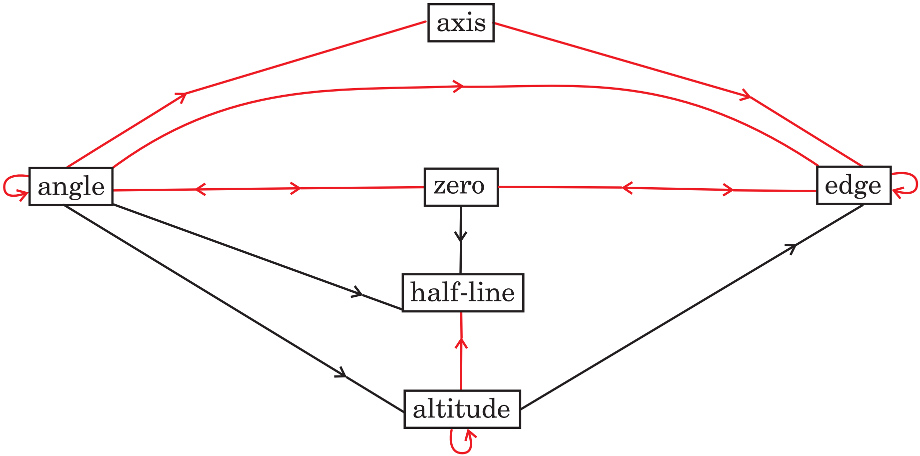
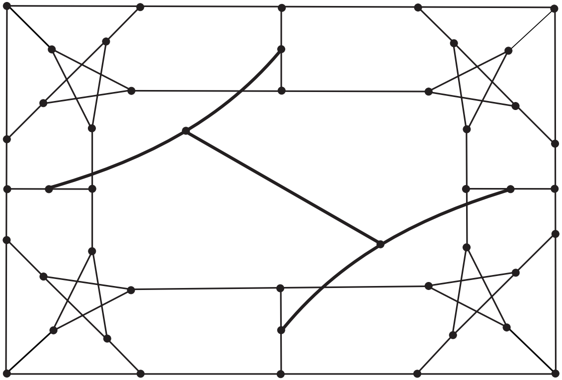


Halin dipoles



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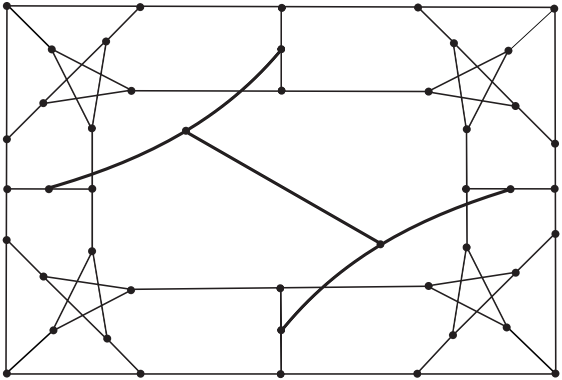
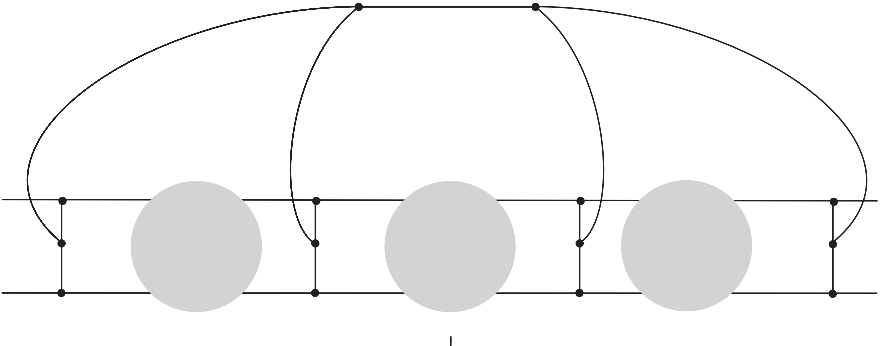


A Halin dipole is a $(2,2)$-pole obtained fom a Halin snark by removing one dislineator.

Halin dipoles

Theorem

Every transition through a Halin dipole has as the form `edge` → `edge` *or* `half-line` → `half-line`.



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A Halin dipole admits only those transitions that are missing in dislineators.

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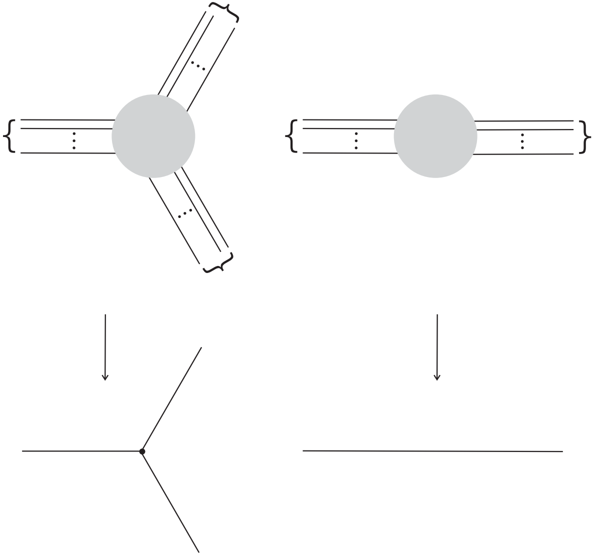
Corollary

Every Halin snark has $\tau \geq 5$.

Superposition



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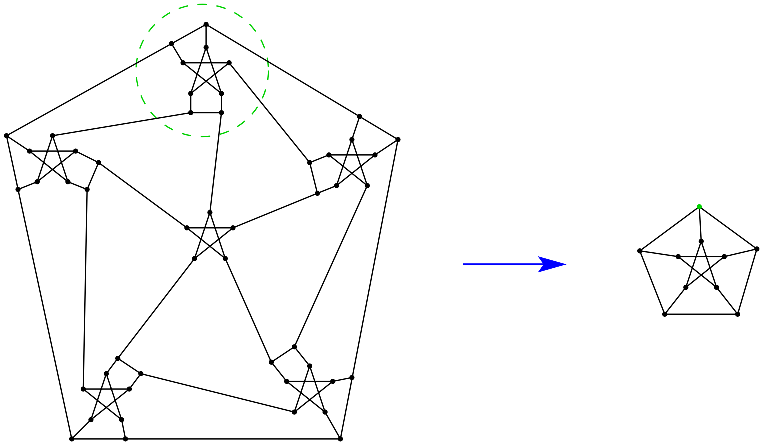

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- every $\mathbb{Z}_2 \times \mathbb{Z}_2$-valuation ϕ of G induces a $\mathbb{Z}_2 \times \mathbb{Z}_2$-valuation ϕ_* of H
- if ϕ is a flow on G, then ϕ_* is a flow on H

(well known from homology theory)

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The aim is to contradict the existence of a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow ϕ on G provided H is a snark.

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- the induced flow ϕ_* is nowhere-zero while H is a snark (contradiction!)

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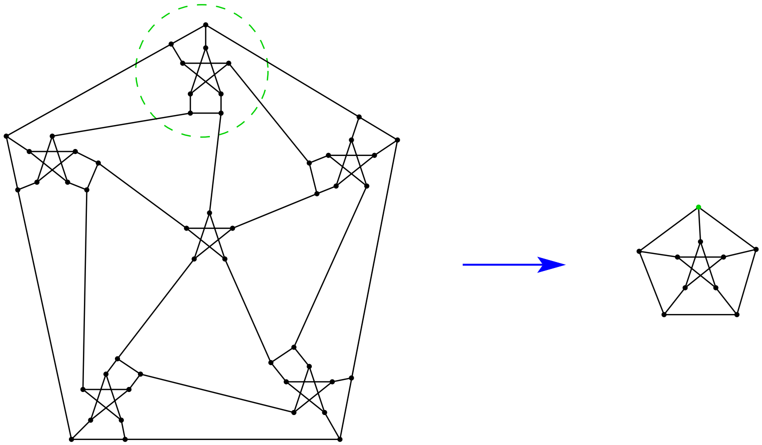
- the induced flow ϕ_* is nowhere-zero while H is a snark (contradiction!)
- the induced valuation ϕ_* fails to be a flow (contradiction!)

Superposition



Graph mappings

Let G and H be graphs.

A graph mapping $f\colon G \to H$ is a mapping from a subdivision G' of G to a subdivision H' of H s.t.

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- edge $\mapsto$ edge or vertex (edge can be contracted to a vertex)
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In this talk:

– G and H will be cubic
– f is onto

Graph mappings, flows, and configurations

- Every $f\colon G \to H$ is continuous if graphs are regarded as 1-dimensional cell complexes. $\Longrightarrow$
- For every abelian group, an A-flow ϕ on G induces an A-flow ϕ_* on H.
- However: ϕ nowhere-zero $\not\Longrightarrow$ ϕ_* nowhere-zero
- We need control over the induced flows $\Longrightarrow$ use configurations

Graph mappings, configurations, and flows.

Let $\mathcal{C} = (P, L)$ be a configuration of points and lines in the n-dimensional projective space $GP(n, 2)$, $n \geq 1$, over the 2-element field. Then

- points ... subset $P \subseteq \mathbb{Z}_2^{n+1} - 0$
- lines ... 3-element subsets $\{x, y, z\}$ of P with $x + y + z = 0$

$\mathcal{C}$-continuous mapping

- Recall that every $\mathcal{C}$-colouring is a special nowhere-zero $\mathbb{Z}_2^{n+1}$-flow.

Definition

A graph mapping $f\colon G \twoheadrightarrow H$ is $\mathcal{C}$-continuous if for every $\mathcal{C}$-flow ϕ on G the induced $\mathbb{Z}_2^{n+1}$-flow ϕ_* is also a $\mathcal{C}$-flow.

$\mathcal{C}$-continuous mappings

Theorem

Let $\mathcal{C}$ be a configuration in $GP(n, 2)$ and let $f\colon G \to H$ be a $\mathcal{C}$-continuous mapping. If H has no $\mathcal{C}$-colouring, then G has no $\mathcal{C}$-colouring.

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We are particularly interested in the cases where $\mathcal{C} = \mathcal{T}$, the tetrahedral configuration in $GP(3,2)$, or $\mathcal{C} = \mathcal{I}$ the trivial configuration consisting of a single projective line.

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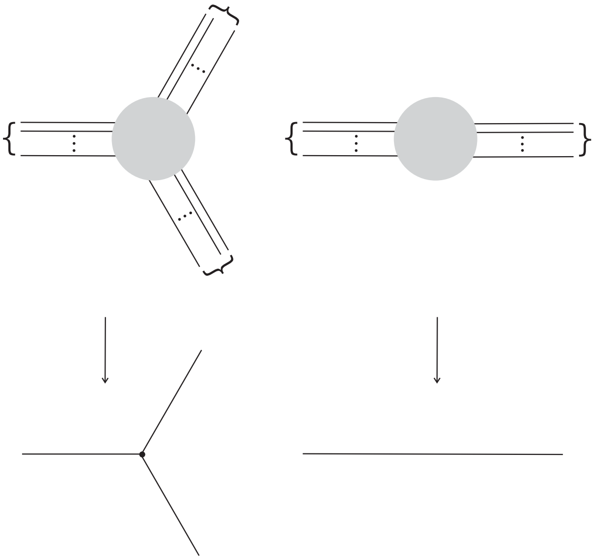
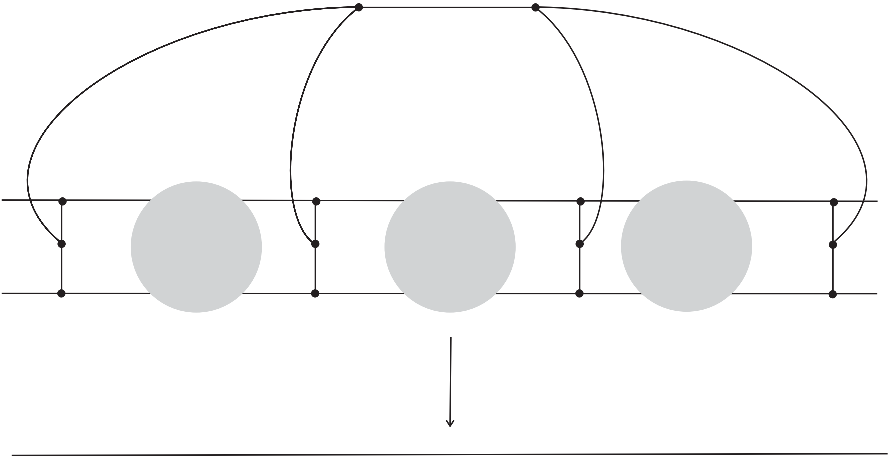
- We can construct large snarks by producing $\mathcal{I}$-continuous mappings to smaller snarks.
- We can construct large snarks with $\tau \geq 5$ by producing $\mathcal{T}$-continuous mappings to smaller snarks with $\tau \geq 5$.

Superposition of snarks with $\tau \geq 5$

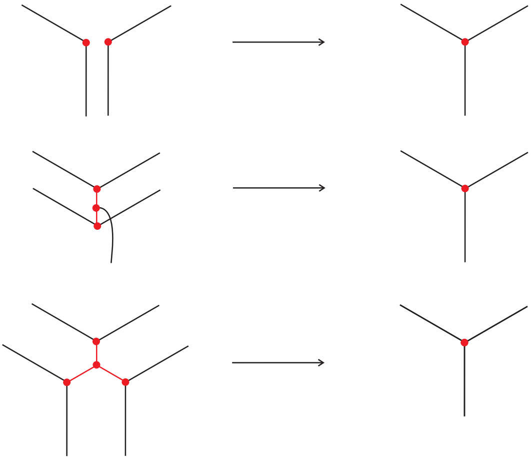


Superedges: Halin dipoles

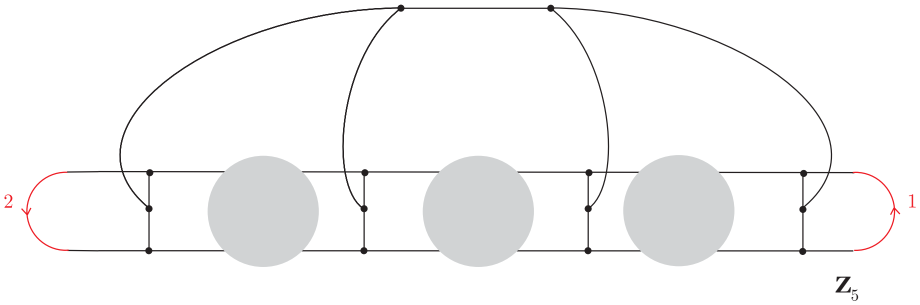


edge → edge   half-line → half-line

Supervertices



Covering contruction



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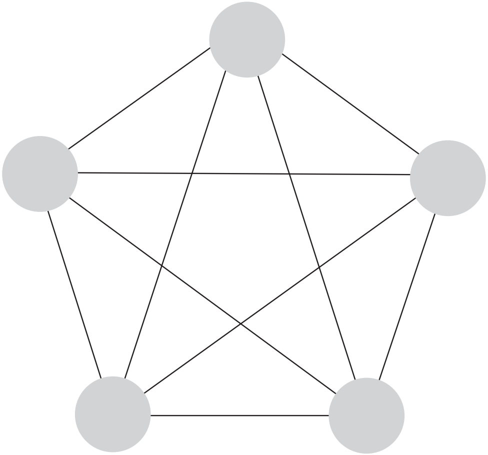

Thank you for your attention