The complexity of surjective homomorphism problems

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Graphs and Homomorphisms.

G, H are graphs; $\mathcal{G}, \mathcal{H}$ are classes of graphs.

A homomorphism from G to H is a function $f : V(G) \to V(H)$ such that, if $(g_1, g_2) \in E(G)$ then $(f(g_1), f(g_2)) \in E(H)$.

The *homomorphism problem* $(\mathcal{G}, \mathcal{H})$-$\textsc{Hom}$ takes as input some $G \in \mathcal{G}, H \in \mathcal{H}$ and asks whether there is a homomorphism from G to H?

$(\mathcal{G}, \mathcal{H})$-Hom, always in NP, has been heavily studied!

The greatest quantity of work considers

$$(-, \{H\})\text{-Hom} = (-, H)\text{-Hom},$$

where the first class is unrestricted and the second singleton. E.g. $(-, K_3)$-Hom is 3-colouring.

Dichotomy Hell and Nešetřil (1990):

- $(-, H)$-Hom is in P if H is bipartite; NP-complete otherwise.

The *constraint satisfaction problem* (CSP) can be considered to be $(-, H)$-HOM, where H is a digraph.

Feder and Vardi conjectured (1993) that

- $(-, H)$-HOM is always in P or is NP-complete.

This contrasts to Ladner's non-dichotomy of NP.

100s of papers have not settled this conjecture! But we live now in exciting times!

One fascinating result in the area states that $(\mathcal{G}, -)$-Hom is in P if

- $\mathcal{G}$ is a digraph class whose cores have bounded treewidth.

Grohe proved, assuming that $\mathrm{FPT} \neq \mathrm{W}[1]$, that these are the only tractable classes $\mathcal{G}$.

So, for $(\mathcal{G}, \mathcal{H})$-Hom: left-hand restrictions well-understood, right-hand restrictions less so.

Many similar problems were studied in the literature.

The problem $(-, H)$-List-Hom takes as input some G together with, for each $g \in G$, lists $l_g \subseteq V(H)$ and asks whether there is a hom f from G to H s.t. $f(g) \subseteq l_g$.

The problem $(-, H)$-Ret takes as input some G containing an induced copy of H and asks whether there is a hom f from G to H that is the identity on H.

Both of these problems are CSPs!

Much is known about the complexity of $(-, H)$-List-Hom.

- classification for (irreflexive) graphs by Feder, Hell and Huang (1999).
  - complement of circular arc in P; rest hard.
- classification for reflexive graphs by Feder and Hell (1998).
  - interval graph in P; rest hard.
- all graphs by Feder, Hell and Huang (2003).
  - bi-arc graphs in P; rest hard.
- full classification by Bulatov (2003).

But the classification of $(-, H)$-Ret, for bipartite H, is known to be as difficult as that of $(-, H)$-Hom for all digraphs!

It is natural to consider other restrictions of homomorphism: injective, surjective and bijective.

- *Injective* is *Subgraph Isomorphism*.
- *Bijective* is *Spanning Subgraph Isomorphism*.

These are both well-studied! Also the edge-surjective variant is studied under the name $(-, H)$-COMP. Indeed, one has:

$$(-, H)\text{-}\textsc{Sur-Hom} \le_{\mathsf{P}}^{\mathrm{Tur}} (-, H)\text{-}\textsc{Comp} \le_{\mathsf{P}}^{\mathrm{Tur}} (-, H)\text{-}\textsc{Ret} \le_{\mathsf{L}} (-, H)\text{-}\textsc{List-Hom}.$$

Locally surjective variant studied - full dichotomy by Fiala and Paulusma (2005).

The problems $(\mathcal{G}, \mathcal{H})$-Sur-Hom have only attracted attention recently.

$(-, C_4^*)$-Sur-Hom attracted much attention in the literature, where it was known variously as *Disconnected Cut*, $2K_2$*-partition* and *Biclique Cover*. It was recently classified as NP-complete.

$(-, H)$-Sur-Hom is NP-complete for non-bipartite H, and may be NP-complete even for some bipartites. A minimal bipartite with this property is announced by Vikas to be C_6

In terms of dichotomy classifications:

Theorem (Golovach, Paulusma and Song 2011)

Let T be a partially reflexive tree. Then, if the vertices in T with a self-loop induce a subtree of T, then $(-, H)$-$\textsc{Sur-Hom}$ is in L. Otherwise, it is NP-complete.

I also mention the following result (away from graphs).

Theorem (Creignou, Khanna and Sudan 2001)

If all relations of B are from one among Horn, dual Horn, bijunctive or affine, $(-, B)$-$\textsc{Sur-Hom}$ is in P. Otherwise, it is NP-complete.

Golovach et al. 2012

$\mathcal{G}$	$\mathcal{H}$	Complexity
complete graphs	all graphs	poly time
all graphs	paths	poly time
paths	all graphs	NP-complete
linear forests	linear forests	NP-complete
unions of cliques	unions of cliques	NP-complete
connected cographs	connected cographs	NP-complete
trees of $\mathrm{pw} \leq 2$	trees of $\mathrm{pw} \leq 2$	NP-complete
split graphs	split graphs	NP-complete
connected proper interval graphs	connected proper interval graphs	NP-complete
trees	trees with k leaves	FPT in k
graphs of $\mathrm{vc} \leq k$	graphs of $\mathrm{vc} \leq k$	FPT in k

New results

Theorem (Feder et al. 2010)

Let H be a connected partially reflexive graph in which the loops do not induce a connected graph. Then $(-, H)$-Ret is NP-complete.

Indeed this is a starting point for the main result of that paper.

Theorem (Feder et al. 2010)

Let H be a partially reflexive pseudoforest. Either $(-, H)$-Ret is in NL or it is NP-complete.

For $(-, H)$-Sur-Hom we cannot do so well.

Theorem (Golovach et al. 2017)

Let H be a connected partially reflexive graph in which the two loops are not adjacent. Then $(-, H)$-Sur-Hom is NP-complete.

Techniques for NP-hardness

Vikas:	reduction from Retraction
M. & Paulusma:	encode a strange problem
Golovach et al.:	reduction from Matching Cut or similar

Future work

Conjecture

Let H be a connected partially reflexive graph in which the loops do not induce a connected graph. Then $(-, H)$-Sur-Hom is NP-complete.

Classifications of $(-, H)$-Sur-Hom?

- On pseudoforests?

Do $(-, H)$-Sur-Hom, $(-, H)$-Comp and $(-, H)$-Ret always have the same complexity?

Theorem (Golovach et al. 2017)

Let H be a partially reflexive graph with at most four vertices. Then $(-, H)$-Sur-Hom, $(-, H)$-Comp and $(-, H)$-Ret have the same complexity

Beyond (di)graphs

3-No-Rainbow-Colouring is $\textsc{Sur-Hom}(B)$ where

$$B := (\{r, b, g\}; \{(x, y, z) : |\{x, y, z\}| < 3\}).$$