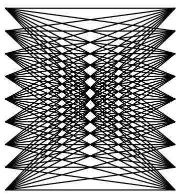
Subgraph counts in large graphs

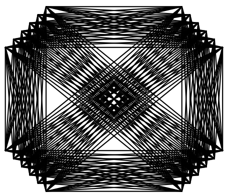
Roman Glebov (Warwick → ETH Zurich → Jerusalem)
Andrzej Grzesik (Krakow → Warsaw → Krakow)
Ping Hu (UIUC → Warwick)
Tamás Hubai (Rényi → Warwick)
Dan Král' (Warwick)
Jan Volec (Warwick → ETH Zurich → McGill)

ATCAGC'17 Durham, 12/1/2017

Introduction

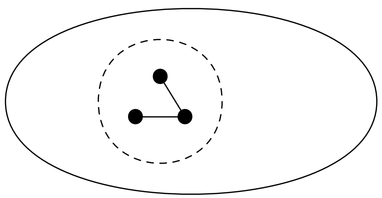
- extremal graph theory
  minimize/maximize graph quantities
- Turán problems
  How many edges can have a graph without K_3? K_5?
  maximum number of edges in an $\mathcal{H}$-free graph
- **What are possible densities of small graphs?**

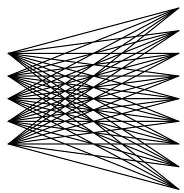


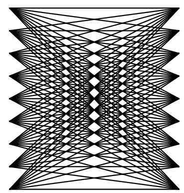


Definitions

- density of H in G
  $d(H,G) =$ probability $|H|$-vertex subgraph of G is H
- feasible densities of $(H_1, \ldots, H_k)$
  $(d_1, \ldots, d_k)$ such that for every $\varepsilon > 0$ and $n \in \mathbb{N}$
  there exists G, $|G| \geq n$, with $|d(H_i, G) - d_i| \leq \varepsilon$
- Theorem (Erdős, Lovász, Spencer, 1979)
  Densities of connected graphs are independent.

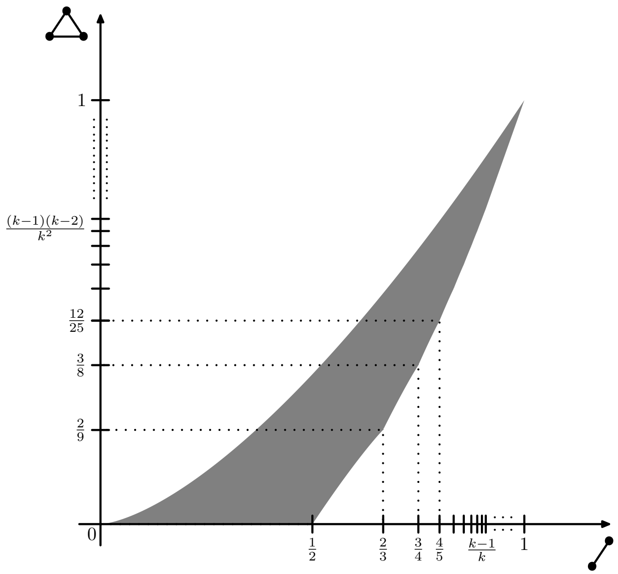






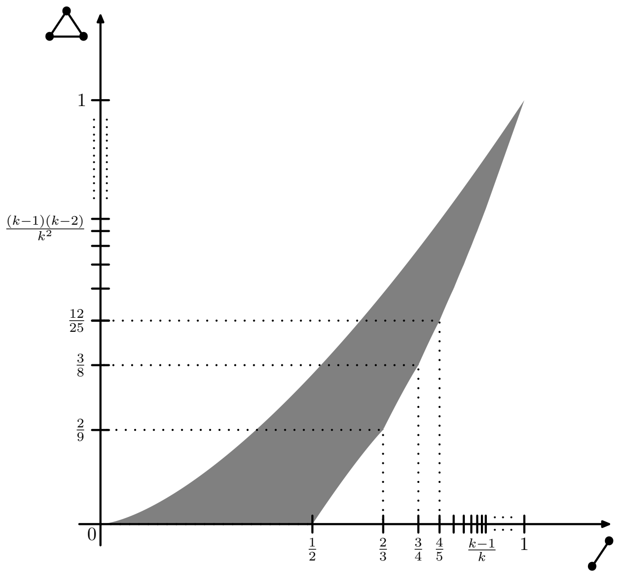
Edge vs. Triangle Problem

- Problem (Erdős, 1955)
  Determine feasible densities of (K_2, K_3).
- upper bound
  complete graph plus isolated vertices



Edge vs. Triangle Problem

- Problem (Erdős, 1955)
  Determine feasible densities of (K_2, K_3).
- lower bound
  complete k-partite graphs with appropriate part sizes

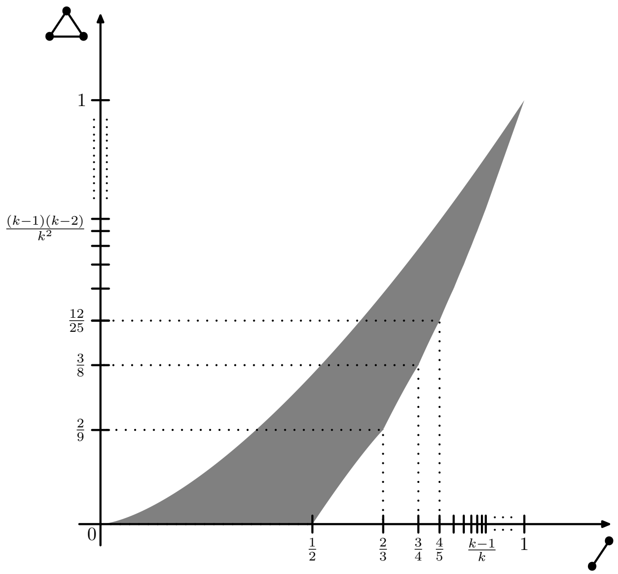


Edge vs. Triangle Problem

- Problem (Erdős, 1955)
  Determine feasible densities of (K_2, K_3).
- Bollobás 1975
  piecewise linear lower bound
- Lovász, Simonovits 1983
  exact lower bound for near non-smooth points
- Fisher 1989
  the first jump (complete 3-partite graphs)

Edge vs. Triangle Problem

- Problem (Erdős, 1955)
  Determine feasible densities of (K_2, K_3).
- solved by Razborov in 2008, flag algebra method
  Pikhurko and Razborov classified extremal graphs



Follow up work

- Problem (Erdős, 1955)
  Determine feasible densities of (K_2, K_3).
- Razborov (2008)
  feasible densities of (K_2, K_3)
- Nikiforov (2011)
  feasible densities of (K_2, K_4)
- Reiher (2016)
  feasible densities of (K_2, K_m)

Graph densities

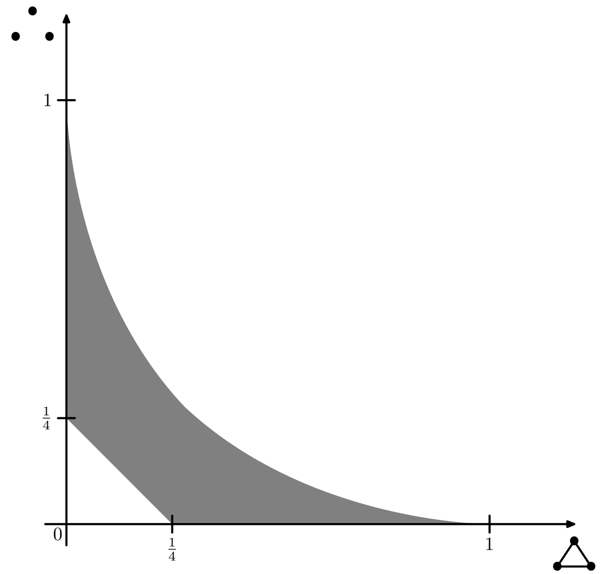
- Theorem (Hatami, Norine, 2011)
  Determining whether $(d_1, \ldots, d_k)$ is feasible
  is undecidable.

- Theorem (Hatami, Norine, 2016+)
  The boundary of the feasible set of densities
  can be nowhere differentiable.

- What about small graphs?
  old problem of Erdős: minimize $\overline{K_4} + K_4$
  Thomason (1989): less than $1/32$

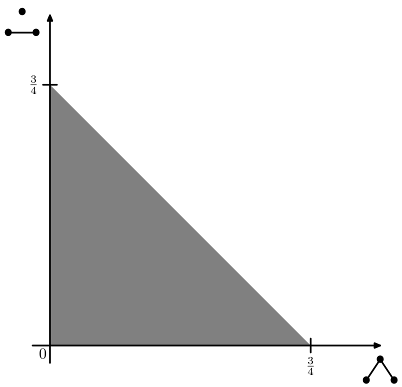
Co-triangle vs. Triangle

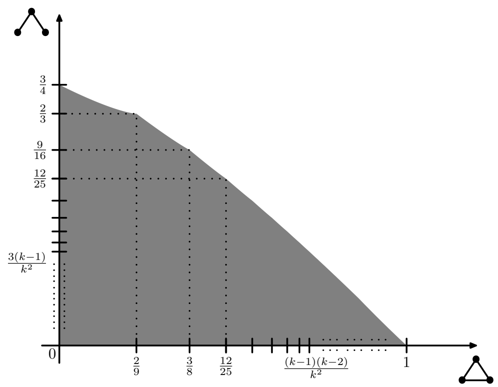
- Theorem (Huang, Linial, Naves, Peled, Sudakov, 2014)
  feasible densities of $(\overline{K_3}, K_3)$
- Goodman's bound: $\overline{K_3} + K_3 \geq 1/4$
  complete graphs plus isolated vertices; complements

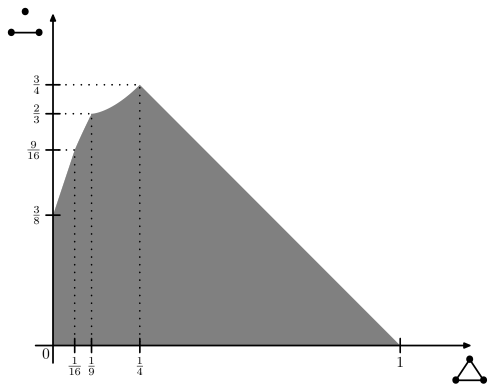


Our results

- Theorem (Glebov, Grzesik, Hu, Hubai, K., Volec)
  feasible densities of all pairs of 3-vertex graphs
  (other pairs are symmetric to those shown below)

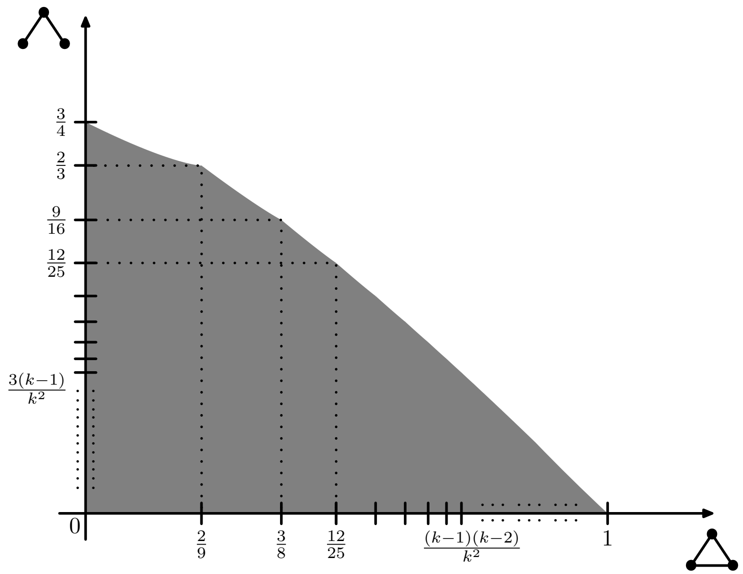






Cherry vs. Triangle

- upper bound attained by complete multipartite graphs
- optimization problem solved using Razborov and Pikhurko-Razborov results



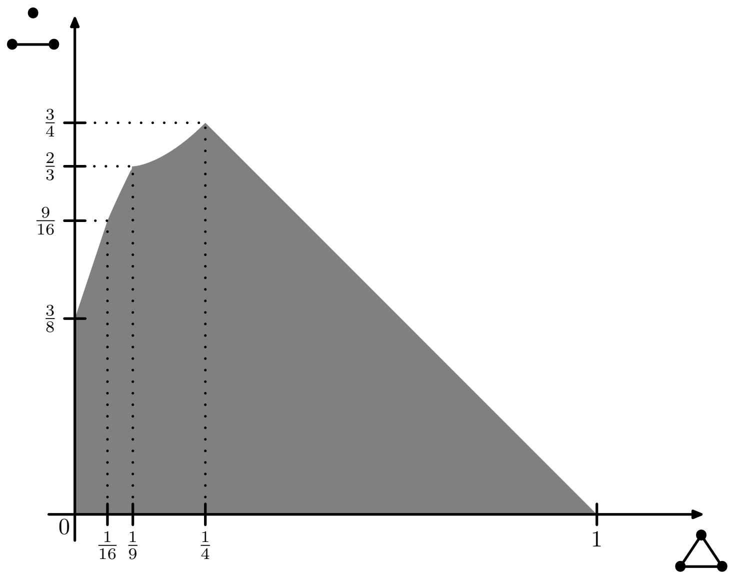
Co-cherry vs. Triangle

- upper bounds

  first: $K_{n,n} \cup K_{n,n} \quad \leftrightarrow \quad K_n \cup K_n \cup K_n \cup K_n$

  second: $K_n \cup K_n \cup \overline{K_{\alpha n,\alpha n}[p]}$ (complex)

  third: $K_n \cup K_n \cup K_{\alpha n}$



Some details of the proof

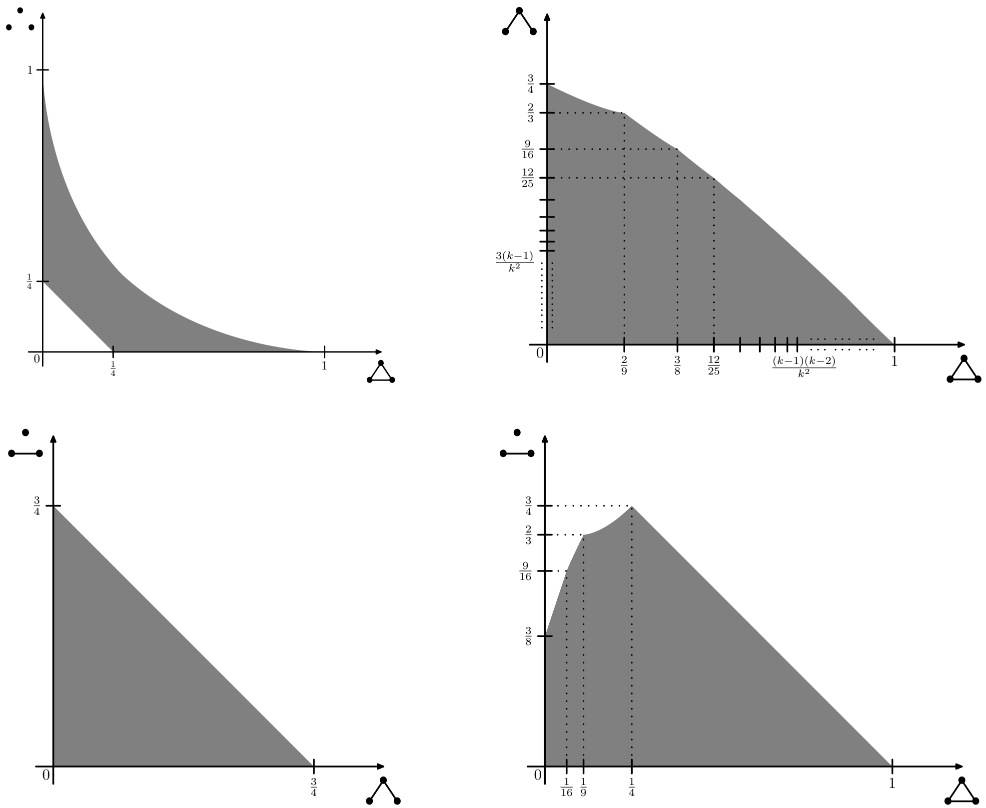
- first curve $(K_{n,n} \cup K_{n,n} \quad \leftrightarrow \quad K_n \cup K_n \cup K_n \cup K_n)$
  computer-free flag algebra calculation

- second curve $(K_n \cup K_n \cup \overline{K_{\alpha n,\alpha n}[p]})$
  maximize $\overline{K_{1,2}} - \alpha K_3$ for $\alpha \in [1,3]$
  differential method to analyze extremal graphs
  component regularity, degrees of non-adjacent vertices
  Lo's edge vs. triangle problem, optimization problem

- third curve $(K_n \cup K_n \cup K_{\alpha n})$
  optimization problem using edge vs. triangle problem

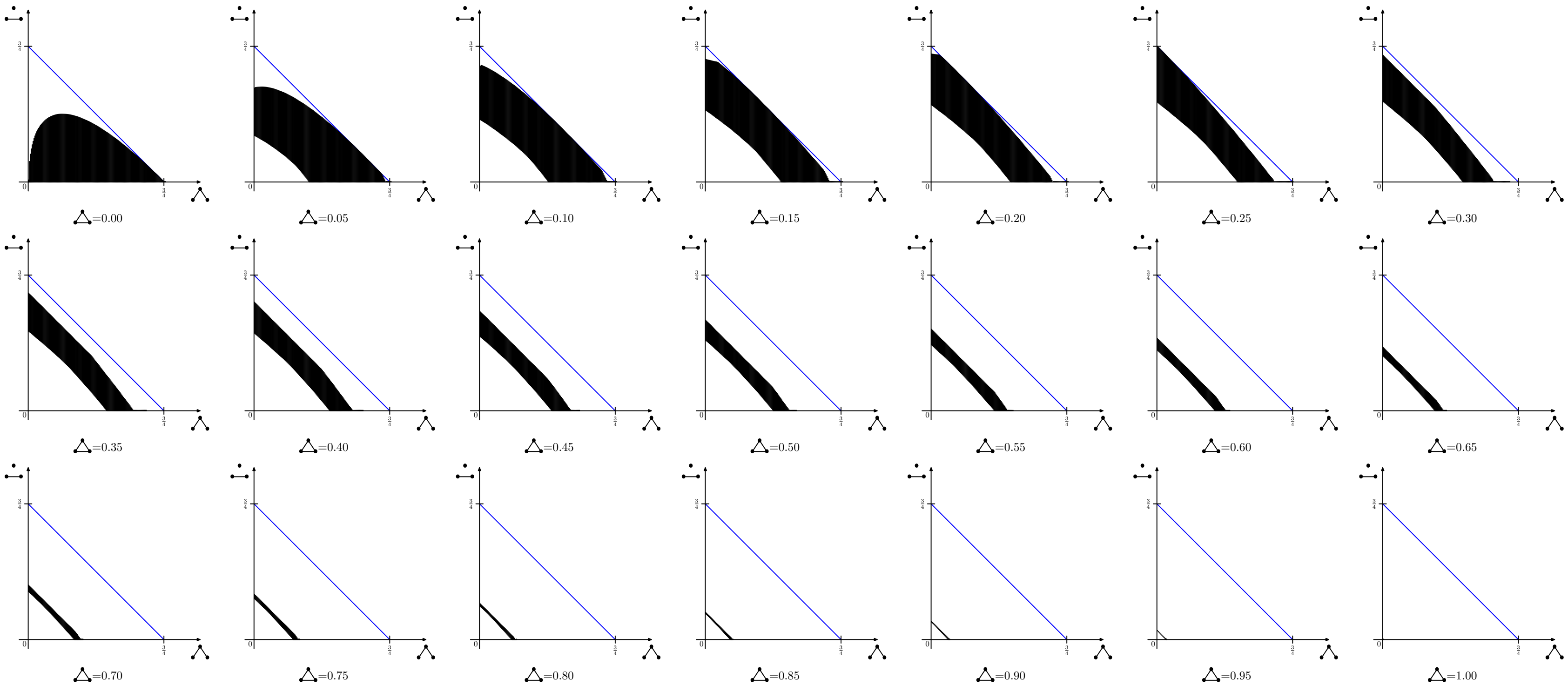
Future work

Determine the 3-dimensional body of $(\overline{K_3}, \overline{K_{1,2}}, K_{1,2}, K_3)$



Future work

Determine the 3-dimensional body of $(\overline{K_3}, \overline{K_{1,2}}, K_{1,2}, K_3)$



Thank you for your attention!